Theoretical Physics
Problem Set 1

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Conventions.

The natural system of units $\hbar = c = 1$ is employed. The flat 4-dimensional Minkowski spacetime $\mathcal{M} \equiv (\mathbb{R}^4, \eta)$ is endowed with the Minkowski metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Indices are raised and lowered using $\eta_{\mu\nu}$ (mapping vectors in $\mathcal{M}$ to covectors in the dual space $\mathcal{M}^*$, and vice versa). The Fourier transform follows the conventions:

$$f(k) = \int \mathrm{d}^4x\, e^{-ik\cdot x} f(x) \qquad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x} f(k) \qquad \delta(y) = \int \frac{\mathrm{d}z}{2\pi}\, e^{-iyz}$$

where $x \in \mathcal{M}$ and $k \in \mathcal{M}^*$, and the latter acts on the former via the canonical pairing $k \cdot x = k_\mu x^\mu$. In the dual space $\mathcal{M}^*$, physical particles are restricted to the *mass shell* 3-dimensional hyperboloid defined by the algebraic constraint $k^2 = m^2$ (with m the particle's mass, which can be zero). The latter condition is also referred to as the *on-shell* condition, while $k^2 \neq m^2$ is called *off-shell*.

Exercise 1. MAXWELL'S THEORY FROM U(1) GAUGE SYMMETRY.

1. Construct the most general action for a real vector field A_μ that is invariant under Lorentz transformations, is at most quadratic in the field, and contains up to two derivatives in total in each term. Omit additive constants and total derivatives.

2. Determine the relations between the coefficients of the action so that it is invariant under the local gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$, for arbitrary $\alpha = \alpha(x)$, at least twice continuously differentiable. Write the result using the antisymmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

3. Add to the action the following term:

$$S_{\text{int}} = -\int \mathrm{d}^4x\, A_\mu J^\mu,$$

where J^μ is an external current. Show that the gauge symmetry requires the current to be conserved, i.e. $\partial_\mu J^\mu = 0$.

4. With the normalization $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, show that the equations of motion in momentum space are written as:

$$D_{\mu\nu}A^\nu(k) = j_\mu(k), \qquad D_{\mu\nu} = -\eta_{\mu\nu}k^2 + k_\mu k_\nu.$$

5. Write $D_{\mu\nu}$ as a 4x4 matrix and show that its determinant is zero. Verify that $D_{\mu\nu}k^\nu = 0$ and explain how the non-invertibility is connected to gauge symmetry.

6. Impose the Lorenz gauge condition, $\partial_\mu A^\mu = 0$, or equivalently $k_\mu A^\mu(k) = 0$. Determine the wave operator that results in the equations of motion in this gauge and invert it for $k^2 \neq 0$. Find the $A^\mu(x)$ produced by the conserved current j^μ.

Solution. We require for the components of the *classical* real vector field $A_\mu : \mathcal{M} \to \mathbb{R}$, where $\mu \in \{0, 1, 2, 3\}$, to be at least twice continuously differentiable, i.e. $A_\mu \in C^2(\mathcal{M})$, as is demanded by the maximum order of derivatives in the action.

1. For the action functional $S[A_\mu] = \int \mathrm{d}^4x\, \mathcal{L}$ to be invariant under Lorentz transformations, the fact that the spacetime measure d^4x is invariant under Lorentz transformations ($|\det(\Lambda)| = 1$, $\forall\, \Lambda \in \mathrm{SO}(1,3)$), implies that the Lagrangian density $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu, ...)$ must be a Lorentz scalar.

Since, by assumption, the Lagrangian density is at most quadratic in the field and contains up to two derivatives in total in each term, Lorentz invariance requires that the Lagrangian density can contain only the following terms (in increasing order in the field A_μ):

- Zeroth Order $O(A^0)$:
  A constant term can be omitted, since it vanishes under variation of the action, while a total derivative term, $\partial_\mu K^\mu(x, A)$, can be omitted since it produces a vanishing surface term in the action. A two-derivative Lorentz scalar term $\Box K(x, A) = \partial_\mu \partial^\mu K(x, A)$ can also be omitted, for the same reason. Therefore, there are no terms of order $O(A^0)$ in the Lagrangian density.
- First Order $O(A^1)$:
  The only possible Lorentz scalar term is $\partial^\mu A_\mu$, which can be omitted for the same reason as above, since it is a total derivative.
- Second Order $O(A^2)$:
  The zero-derivative Lorentz scalar term $A_\mu A^\mu$ is allowed, while there are no one-derivative Lorentz scalar terms. The possible two-derivative Lorentz scalar terms are $\partial_\mu A_\nu \partial^\mu A^\nu$, $\partial_\mu A_\nu \partial^\nu A^\mu$, $(\partial_\mu A^\mu)(\partial_\nu A^\nu)$, $A_\mu \Box A^\mu$, and $A_\mu \partial^\mu \partial^\nu A_\nu$. Because of the following identities (integration by parts):

  $$\begin{aligned} A_\nu \Box A^\nu &= \partial_\mu (A_\nu \partial^\mu A^\nu) - \partial_\mu A_\nu \partial^\mu A^\nu \\ A_\nu \partial_\mu \partial^\nu A^\mu &= \partial_\mu (A_\nu \partial^\nu A^\mu) - \partial_\mu A_\nu \partial^\nu A^\mu \\ (\partial_\mu A_\mu)(\partial^\nu A^\nu) &= \partial_\mu (A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu) + \partial_\mu A_\nu \partial^\nu A^\mu \end{aligned}$$

  and the fact that total derivative terms produce vanishing surface terms in the action, we can reduce the number of independent two-derivative Lorentz scalar terms to just two, i.e. $\partial_\mu A_\nu \partial^\mu A^\nu$ and $\partial_\mu A_\nu \partial^\nu A^\mu$.

Therefore, the most general action for a real vector field A_μ that is invariant under Lorentz transformations, is at most quadratic in the field, and contains up to two derivatives in total in each term, is given by:

$$S = \int \mathrm{d}^4 x \, \mathcal{L}(A_\mu, \partial_\nu A_\mu) \tag{1}$$

$$\boxed{\mathcal{L}(A_\mu, \partial_\nu A_\mu) = \alpha \cdot \partial_\mu A_\nu \partial^\mu A^\nu + \beta \cdot \partial_\mu A_\nu \partial^\nu A^\mu + \gamma \cdot A_\mu A^\mu} \tag{2}$$

for some real coefficients $\alpha, \beta, \gamma \in \mathbb{R}$. Note that, in general, the Lagrangian density $\mathcal{L}$ is a function of the field A_μ, its first $\partial_\nu A_\mu$, and higher order derivatives, but Lorentz invariance and the restriction to at most two derivatives in total in each term (along with integration by parts), eliminates the latter, so that $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu)$.

2. Under the dictated local gauge transformation, the field A_μ and its derivative $\partial_\nu A_\mu$ transform as:

$$\begin{aligned} A_\mu &\to A'_\mu = A_\mu + \partial_\mu \alpha \\ \partial_\mu A_\nu &\to \partial_\mu A'_\nu = \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha \end{aligned}$$

Substitution into the Lagrangian density (2) gives:

$$\begin{aligned} \mathcal{L}' = &\alpha \cdot (\partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha)(\partial^\mu A^\nu + \partial^\mu \partial^\nu \alpha) \\ &+ \beta \cdot (\partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha)(\partial^\nu A^\mu + \partial^\nu \partial^\mu \alpha) \\ &+ \gamma \cdot (A_\mu + \partial_\mu \alpha)(A^\mu + \partial^\mu \alpha) \end{aligned}$$

which, employing the symmetry of mixed partial derivatives ($\alpha \in C^2(\mathcal{M})$), i.e. $\partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha$, can be expanded to give:

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$$

where $\delta\mathcal{L}$ is the change in the Lagrangian density under the local gauge transformation, given by:

$$\begin{aligned} \delta\mathcal{L} = &2(\alpha + \beta)(\partial_\mu A_\nu)(\partial^\mu \partial^\nu \alpha) + 2\gamma A_\mu \partial^\mu \alpha \\ &+ (\alpha + \beta)(\partial_\mu \partial_\nu \alpha)(\partial^\mu \partial^\nu \alpha) + \gamma(\partial_\mu \alpha)(\partial^\mu \alpha) \end{aligned}$$

For the action functional to be invariant under the local gauge transformation, $\delta\mathcal{L}$ is required to be a total derivative (corresponding to a quasi-symmetry of the Lagrangian density), so that it integrates to a vanishing boundary term, for any arbitrary field configuration $A_\mu(x)$ and any arbitrary gauge parameter $\alpha(x)$. The latter condition is equivalent to requiring that $\delta\mathcal{L}$ satisfies the Euler-Lagrange equations for the field A_μ and the gauge parameter α identically, i.e.:

$$E_{A_\mu}(\delta\mathcal{L}) = 0, \qquad E_\alpha(\delta\mathcal{L}) = 0, \qquad \forall\, A_\mu(x), \alpha(x) \in C^2(\mathcal{M})$$

where:

$$E_{A_\mu} = \frac{\partial}{\partial A_\mu} - \partial_\nu \left(\frac{\partial}{\partial(\partial_\nu A_\mu)} \right), \qquad E_\alpha = \frac{\partial}{\partial \alpha} - \partial_\mu \left(\frac{\partial}{\partial(\partial_\mu \alpha)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial}{\partial(\partial_\mu \partial_\nu \alpha)} \right)$$

are the Euler-Lagrange operators for the field A_μ and the gauge parameter α (which dynamically appears in the Lagrangian variation up to a second derivative), respectively. Simple calculation gives:

$$\begin{aligned} E_{A_\mu}(\delta\mathcal{L}) &= 2\gamma \partial^\mu \alpha - 2(\alpha + \beta)\Box \partial^\mu \alpha = 0 \\ E_\alpha(\delta\mathcal{L}) &= -2\gamma(\partial_\mu A^\mu + \Box\alpha) + 2(\alpha + \beta)\Box(\partial_\mu A^\mu + \Box\alpha) = 0 \end{aligned}$$

for all $A_\mu(x)$ and $\alpha(x) \in C^2(\mathcal{M})$, from which the relations between the coefficients of the action that are required for invariance under the local gauge transformation can immediately be read off:

$$\gamma = 0 \qquad \text{and} \qquad \alpha + \beta = 0 \tag{3}$$

Applying these relations to the Lagrangian density (2), and employing the symmetry of mixed partial derivatives, the gauge-invariant Lagrangian density can be written as:

$$S = \int \mathrm{d}^4x \, \mathcal{L}(A_\mu, \partial_\nu A_\mu) \tag{4}$$

$$\boxed{\mathcal{L}(A_\mu, \partial_\nu A_\mu) = \frac{\alpha}{2} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{5}$$

for some real coefficient $\alpha \in \mathbb{R}$, thus recovering the Maxwell Lagrangian from requiring a U(1) local gauge symmetry of the action.

3. Adding the interaction term S_{int} to the action, the total action and Lagrangian density become:

$$\begin{aligned} S_{\text{total}} &= S + S_{\text{int}} = S_{\text{int}} - \int \mathrm{d}^4x A_\mu j^\mu \\ \mathcal{L}_{\text{total}} &= \mathcal{L} + \mathcal{L}_{\text{int}} = \mathcal{L} - A_\mu j^\mu \end{aligned}$$

For the total action to be invariant under the local gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$, the variation of the interaction term Lagrangian must be a total derivative. Since:

$$\begin{aligned} \delta \mathcal{L}_{\text{int}} &= -\delta(A_\mu j^\mu) = -(\partial_\mu \alpha) j^\mu \\ &= -\partial_\mu(\alpha j^\mu) + \alpha(\partial_\mu j^\mu), \qquad \forall\, \alpha(x) \in C^2(\mathcal{M}) \end{aligned}$$

it is necessary and sufficient for the current to be conserved, i.e. $\partial_\mu j^\mu = 0$ for the variation of the interaction term Lagrangian to be a total derivative. Equivalently, and perhaps more rigorously, the variation of the interaction S_{int} must vanish identically for any arbitrary gauge parameter $\alpha(x)$. Since:

$$\begin{aligned} \delta S_{\text{total}} = \delta S_{\text{int}} &= -\int \mathrm{d}^4x \, (\partial_\mu \alpha) j^\mu \\ &= -\int \mathrm{d}^4x \, \partial_\mu(\alpha j^\mu) + \int \mathrm{d}^4x \alpha(\partial_\mu j^\mu), \qquad \forall\, \alpha(x) \in C^2(\mathcal{M}) \end{aligned}$$

restricting the gauge parameter $\alpha(x)$ to the subspace of $C^2(\mathcal{M})$ of functions with compact support, $\alpha(x) \in C_c^2(\mathcal{M})$, the first term vanishes by the divergence theorem, and by the fundamental lemma of the calculus of variations,

the second term vanishes identically for any arbitrary $\alpha(x) \in C_c^2(\mathcal{M})$ if, and only if, the current is conserved, i.e.:

$$\boxed{\partial_\mu j^\mu = 0} \tag{6}$$

Identity (6) is the continuity equation and, being a direct consequence of the local gauge symmetry of the action, it is an *off-shell* Noether identity (resulting from Noether's Second Theorem) that holds for any arbitrary field configuration $A_\mu(x)$, and is *not* a consequence of the equations of motion.

4. The normalized total action functional and total Lagrangian density (hereby denoted without the subscript) are given by:

$$S = \int \mathrm{d}^4x \, \mathcal{L}(A_\mu, \partial_\nu A_\mu) \tag{7}$$

$$\boxed{\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu} \tag{8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric field strength tensor. The equations of motion for the field A_μ are obtained easily from the Euler-Lagrange equations:

$$\boxed{\partial_\nu F^{\nu\mu} = j^\mu \quad \Leftrightarrow \quad \Box A_\mu - \partial_\mu \partial_\nu A^\nu = j_\mu} \tag{9}$$

where the second form is obtained by substituting the definition of the field strength tensor $F_{\mu\nu}$ into the first form and lowering indices via the metric tensor $\eta_{\mu\nu}$.
The equations of motion in momentum space are obtained by Fourier transforming the field A_μ and the current j_μ (cf. Conventions):

$$A_\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \, e^{ik\cdot x} A_\mu(k)$$
$$j_\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \, e^{ik\cdot x} j_\mu(k)$$

and substituting into the equations of motion (9), giving:

$$\Box \left(\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ikx} A_\mu(k) \right) - \partial_\mu \partial_\nu \left(\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ikx} A^\nu(k) \right) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ikx} j_\mu(k)$$

Since $A_\mu(k)$ and $j_\mu(k)$ are functions of $k \in \mathcal{M}^*$ that belong in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$, the Lebesgue Dominated Convergence theorem allows for the interchange of the order of differentiation and integration, giving:

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} (-k^2 e^{ikx}) A_\mu(k) - \int \frac{\mathrm{d}^4k}{(2\pi)^4} (-k_\mu k_\nu e^{ikx}) A^\nu(k) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ikx} j_\mu(k)$$

which simplifies to:

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ikx} \left[-k^2 A_\mu(k) + k_\mu k_\nu A^\nu(k) - j_\mu(k)\right] = 0, \quad \forall\, x \in \mathcal{M}$$

For the latter to hold for any arbitrary $x \in \mathcal{M}$, the integrand must vanish identically, giving the equations of motion in momentum space:

$$\boxed{D_{\mu\nu} A^\nu(k) = j_\mu(k), \qquad D_{\mu\nu} \coloneqq -\eta_{\mu\nu} k^2 + k_\mu k_\nu} \tag{10}$$

5. Choosing a basis of $\mathcal{M}^*$ in which $k_\mu = (k_0, \mathbf{k}) = (k_0, k_1, k_2, k_3)$, thereby $k^2 = k_0^2 - \mathbf{k}^2$, and in which the Minkowski metric takes its standard form, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, the differential operator in (10) is written in matrix form as:

$$D \coloneqq (D_{\mu\nu}) = \begin{pmatrix} -k^2 + k_0^2 & k_0 k_1 & k_0 k_2 & k_0 k_3 \\ k_1 k_0 & k^2 + k_1^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_0 & k_2 k_1 & k^2 + k_2^2 & k_2 k_3 \\ k_3 k_0 & k_3 k_1 & k_3 k_2 & k^2 + k_3^2 \end{pmatrix}$$

The vanishing determinant of the above matrix can be computed directly, or by noting that the matrix is of the form $D = -k^2\eta + kk^T$, where η is the Minkowski metric and $k \equiv (k_0, k_1, k_2, k_3)^T$ is treated as a column vector (via the obvious isomorphism of $\mathcal{M}^*$ with $\mathbb{R}^4$). The determinant can then be computed using the Matrix Determinant Lemma, which states that for an invertible matrix A and vectors u and v, $\det(A + uv^T) = (1 + v^T A^{-1} u)\det(A)$. In this case, since $-k^2\eta$ is invertible for $k^2 \neq 0$, it applies that:

$$\begin{aligned} \det(D) &= \det(-k^2\eta + kk^T) = \det(-k^2\eta) \cdot (1 + k^T(-k^2\eta)^{-1}k) \\ &= -(k^2)^4 \cdot (1 - \frac{k^T\eta k}{k^2}) = -(k^2)^4 \cdot (1 - \frac{k^2}{k^2}) = 0 \end{aligned}$$

hence:

$$\boxed{\det(D) = \det(D_{\mu\nu}) = 0} \tag{11}$$

and, since the last equality holds for any choice of basis of $\mathcal{M}^*$ (as it can easily be seen via a change of basis), the wave operator $D_{\mu\nu}$ is non-invertible. Explicitly computing the product of the wave operator with the momentum vector k gives:

$$\begin{aligned} D_{\mu\nu} k^\nu &= (-\eta_{\mu\nu} k^2 + k_\mu k_\nu) k^\nu = -\eta_{\mu\nu} k^2 k^\nu + k_\mu (k_\nu k^\nu) \\ &= -k^2 k_\mu + k^2 k_\mu = 0 \end{aligned}$$

hence:

$$\boxed{D_{\mu\nu}k^\nu = 0 \;\Leftrightarrow\; \ker\{D_{\mu\nu}\} \neq \varnothing} \tag{12}$$

From the fact that $D = k^2 I + k \otimes k$ it can be immediately deduced that also:

$$\boxed{\dim\ker\{D_{\mu\nu}\} = 1 \;\Leftrightarrow\; \operatorname{rank}\{D_{\mu\nu}\} = 3} \tag{13}$$

The fact that the wave operator $D_{\mu\nu}$ has a non-empty kernel (i.e. it is non-invertible) is a direct consequence of the local gauge symmetry of the action. Indeed, requiring the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\alpha$ expressed in momentum space, $A_\mu(k) \rightarrow A_\mu(k) + ik_\mu\alpha(k)$, to be a local symmetry of the equations of motion (10), it can be seen that:

$$\begin{aligned}
& D_{\mu\nu}(A^\nu(k) + ik^\nu\alpha(k)) = j_\mu(k) \\
\Leftrightarrow\quad & D_{\mu\nu}A^\nu(k) + iD_{\mu\nu}k^\nu\alpha(k) = j_\mu(k) \\
\Leftrightarrow\quad & D_{\mu\nu}k^\nu\alpha(k) = 0, \qquad \forall\,\alpha(k) \in C^2(\mathcal{M}^*)
\end{aligned}$$

which is satisfied identically for any arbitrary $\alpha(k)$ if, and only if, $D_{\mu\nu}k^\nu = 0$. Conversely, had the wave operator $D_{\mu\nu}$ been invertible, the local gauge symmetry would have been broken, since the only solution to the equation $D_{\mu\nu}k^\nu\alpha(k) = 0$ would have been the trivial solution $\alpha(k) = 0$, which is not a local gauge transformation, and the equations of motion (10) would have had a *unique* solution for the field $A_\mu(k)$.

To summarize, the U(1) *local* gauge symmetry of the general action (1) for the real vector field A_μ is equivalent to the antisymmetry of the field strength tensor $F_{\mu\nu}$, which is equivalent to the non-dynamical nature of the A_0 component of the field, i.e. the canonical momentum conjugate to the temporal component is identically zero:

$$\pi^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = -F^{00} = 0,$$

which is equivalent to the non-invertibility of the wave operator $D_{\mu\nu}$, which is equivalent to the fact that longitudinal modes of the field A_μ are not dynamical, i.e. they do not correspond to propagating degrees of freedom. Hence, the local gauge symmetry of the action is equivalent to the fact that only the two transverse modes of the field A_μ are dynamical, which is the physical content of Maxwell's theory for a massless vector field (such as the photon in electromagnetism).

6. Imposing the Lorenz gauge condition, $\partial_\mu A^\mu = 0$, or equivalently in momentum space, $k_\mu A^\mu(k) = 0$, the equations of motion (10) reduce to:

$$\boxed{\Box A_\mu(x) = j_\mu(x) \quad \Leftrightarrow \quad -\eta_{\mu\nu} k^2 A^\nu(k) = j_\mu(k)} \tag{14}$$

Thus, the wave operator in the Lorenz gauge is simply the diagonal part of the original wave operator, i.e. $\tilde{D}_{\mu\nu} \coloneqq -\eta_{\mu\nu} k^2$, which is invertible for $k^2 \neq 0$, with inverse:

$$\boxed{\tilde{D}^{-1\,\mu\nu} = -\frac{\eta^{\mu\nu}}{k^2}, \quad k^2 \neq 0} \tag{15}$$

The solution for the field $A^\mu(k)$ in momentum space is then given by:

$$\boxed{A^\mu(k) = \tilde{D}^{-1\,\mu\nu} j_\nu(k) = -\frac{j^\mu(k)}{k^2}, \quad k^2 \neq 0} \tag{16}$$

The solution for the field $A^\mu(x)$ in spacetime is obtained by inverse Fourier transforming (16):

$$\boxed{A^\mu(x) = -\int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{ikx} \frac{j^\mu(k)}{k^2}, \quad k^2 \neq 0} \tag{17}$$

which is the solution for the field $A^\mu(x)$ produced by the (locally) conserved current $j^\mu(x)$ in the Lorenz gauge.

The integral in (17) is singular at $k^2 = 0$ (on the *mass-shell*, i.e. the light cone), which is a manifestation of the fact that the Lorenz gauge condition does *not* completely fix the gauge, since there is still a residual gauge freedom for gauge parameters $\alpha(x)$ that satisfy the wave equation $\Box\, \alpha(x) = 0$.

The evaluation of the integral in (17) first involves substitution of the Fourier transform of the current, yielding:

$$A^\mu(x) = \int \mathrm{d}^4 y\, G(x - y) j^\mu(y) \tag{18}$$

where:

$$G(x - y) \equiv -\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2} \tag{19}$$

is the Green's function for the d'Alembertian operator. The integration of the diverging temporal component ($\mathrm{d}k_0$) in (19) requires the analytic continuation of the integrand function in the complex k_0-plane, and, demanding the

dynamics of the fields to *preserve causality*, the choice of a closed Jordan contour that, for $t > 0$, shifts the poles in the upper-half plane and closes in the lower-half plane. An application of Cauchy's Residue theorem, and an additional 3-dimensional integration ($\mathrm{d}^3\mathbf{k}$) yields for the Green function:

$$\boxed{G(x-y) = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|}\delta\left((x^0-y^0) - |\mathbf{x}-\mathbf{y}|\right)} \tag{20}$$

Substitution in (18) gives:

$$A^\mu(t,\mathbf{x}) = \int \mathrm{d}t'\,\mathrm{d}^3\mathbf{y}\,\frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|}\delta(t-t'-|\mathbf{x}-\mathbf{y}|)\,j^\mu(t',\mathbf{y})$$

which, after integrating the δ-distribution, gives the final result:

$$\boxed{A^\mu(t,\mathbf{x}) = \int \mathrm{d}^3\mathbf{y}\,\frac{j^\mu(t-|\mathbf{x}-\mathbf{y}|\,,\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|}} \tag{21}$$

The result (21) is the Retarded Potential formulae, explicitly stating that calculation of the field at position $\mathbf{x}$ and time t, is the sum of the contributions of the current j^μ at all other spatial points $\mathbf{y}$, evaluated at a "retarded time" $t_{\mathrm{ret}} = t - r$ (where $r = |\mathbf{x}-\mathbf{y}|$ is the corresponding distance).

■

Exercise 2. LINEARIZED GRAVITY AND THE SPIN-2 FIELD.

The most general action for a real, symmetric tensor field $h_{\mu\nu} = h_{\nu\mu}$, which is quadratic in the field, contains up to two derivatives in total in each term, and is invariant under Lorentz transformations and under the local gauge transformations

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \xi_\mu = \xi_\mu(x),$$

is given by:

$$S = -\frac{1}{16\pi G_N} \int \mathrm{d}^4x \left[\partial_\mu h^{\kappa\nu} \partial^\mu h_{\kappa\nu} + 2\partial_\mu h^{\mu\nu} \partial_\nu h - 2\partial_\mu h^{\kappa\mu} \partial^\nu h_{\kappa\nu} - \partial_\mu h \partial^\mu h \right],$$

where $h = \eta^{\mu\nu} h_{\mu\nu} \equiv h^\kappa{}_\kappa$ and G_N is Newton's constant.

1. Add to the action the term $S_{\text{int}} = \int \mathrm{d}^4x \, h_{\mu\nu} T^{\mu\nu}$, where $T^{\mu\nu}$ is the symmetric energy-momentum tensor of an external source. Show that invariance under the aforementioned gauge transformations requires $\partial_\mu T^{\mu\nu} = 0$.

2. Find the equations of motion of the field in the presence of the source.

3. Show that in the harmonic gauge, $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$, which is analogous to the Lorenz gauge in electrodynamics (cf. Exercise 1), the equations of motion become:
$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \Box h = -8\pi G_N T_{\mu\nu}.$$

4. Write the above equations in momentum space as:
$$D_{\mu\nu\alpha\beta} h^{\alpha\beta}(k) = -8\pi G_N T_{\mu\nu}(k),$$
and determine the wave operator $D_{\mu\nu\alpha\beta}$.

5. Using the relation:
$$P^{\mu\nu\alpha\beta} D_{\alpha\beta\rho\sigma} = \frac{1}{2}(\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\sigma \delta^\nu{}_\rho),$$
find the inverse $P^{\mu\nu\alpha\beta}$ of the wave operator for $k^2 \neq 0$.

6. Using the previous results, show that the solution produced by the source is written as:
$$h_{\mu\nu}(x) = 8\pi G_N \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2} \left[T_{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu} T^\rho{}_\rho(k) \right].$$

7. Consider that the source corresponds to a point mass M located at rest at the position $\mathbf{x} = 0$, i.e. the energy-momentum tensor is given by:

$$T_{\mu\nu}(\mathbf{x}) = M\,\delta^0{}_\mu\delta^0{}_\nu\delta^{(3)}(\mathbf{x}),$$

where $\delta^{(3)}(\mathbf{x})$ is the 3-dimensional Dirac delta distribution. Calculate $T_{\mu\nu}(k)$ and substitute it into the expression for $h_{\mu\nu}(x)$. For the static solution that vanishes at spatial infinity, show that:

$$h_{00}(x) = \phi \qquad h_{0i}(x) = 0 \qquad h_{ij}(x) = \phi\,\delta_{ij}, \quad i,j \in \{1,2,3\}$$

where $\phi = \phi(r)$ is the Newtonian potential, given by $\phi = \phi(r) = -\frac{G_N M}{r}$, and $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2} > 0$.

8. Consider now that $h_{\mu\nu}$ is a small perturbation of the Minkowski metric, with the normalization $g_{\mu\nu} \simeq \eta_{\mu\nu} + 2h_{\mu\nu}$. That is, for the solution of the previous question it holds that:

$$g_{00} \simeq 1 + 2h_{00} \qquad g_{0i} = g_{i0} = 0 \qquad g_{ij} \simeq -\delta_{ij} + 2h_{ij}, \quad i,j \in \{1,2,3\}$$

Write the infinitesimal line element corresponding to this metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

in Cartesian coordinates (t,x,y,z) and spherical coordinates (t,r,θ,φ).

9. Find an appropriate transformation of the radial coordinate, i.e. find a suitable $R = R(r)$, so that the line element, to first order in $G_N M/r$, coincides with that of the Schwarzschild geometry in the weak-field approximation:

$$\mathrm{d}s^2_{\mathrm{Sch}} \simeq \left(1 - \frac{2G_N M}{R}\right)\mathrm{d}t^2 - \left(1 + \frac{2G_N M}{R}\right)\mathrm{d}R^2 - R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2).$$

Solution. We require for the *classical* real, symmetric tensor field $h_{\mu\nu}\colon \mathcal{M} \to \mathbb{R}$, where $\mu,\nu \in \{0,1,2,3\}$ and $h_{\mu\nu} = h_{\nu\mu}$, to be at least twice continuously differentiable, i.e. $h_{\mu\nu} \in C^2(\mathcal{M})$, as is demanded by the maximum order of derivatives present in the free action. Furthermore, the local gauge parameter vector field $\xi_\mu\colon \mathcal{M} \to \mathbb{R}$ is required to be at least three times continuously differentiable, i.e. $\xi_\mu \in C^3(\mathcal{M})$, to ensure that the gauge-transformed field $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ strictly remains in the $C^2(\mathcal{M})$ function space.

1. By the premise of the problem, the free action S is invariant under the local gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, meaning its variation

vanishes identically: $\delta S = 0$. Therefore, for the total action $S_{\text{total}} = S + S_{\text{int}}$ to be gauge invariant, it is necessary and sufficient that the variation of the interaction term vanishes up to a total derivative, thereby producing a zero boundary term.

The variation of the symmetric tensor field is strictly:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Substituting this into the variation of the interaction action gives:

$$\begin{aligned}
\delta S_{\text{int}} &= \int \mathrm{d}^4x \, (\delta h_{\mu\nu}) T^{\mu\nu} \\
&= \int \mathrm{d}^4x \, (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) T^{\mu\nu} \\
&= \int \mathrm{d}^4x \, (\partial_\mu \xi_\nu) T^{\mu\nu} + \int \mathrm{d}^4x \, (\partial_\nu \xi_\mu) T^{\mu\nu}
\end{aligned}$$

Because the energy-momentum tensor of the external source is (or, is *made* to be) symmetric, i.e. $T^{\mu\nu} = T^{\nu\mu}$, renaming the dummy integration indices in the second integral ($\mu \leftrightarrow \nu$) shows that the two terms are identical:

$$\delta S_{\text{int}} = 2 \int \mathrm{d}^4x \, (\partial_\mu \xi_\nu) T^{\mu\nu}$$

Applying integration by parts, it can be seen that:

$$\delta S_{\text{int}} = 2 \int \mathrm{d}^4x \, \partial_\mu (\xi_\nu T^{\mu\nu}) - 2 \int \mathrm{d}^4x \, \xi_\nu (\partial_\mu T^{\mu\nu})$$

Restricting the arbitrary gauge parameter field $\xi_\nu(x)$ to the subspace of $C^3(\mathcal{M})$ of functions with compact support, $\xi_\nu \in C^3_c(\mathcal{M})$, the first integral transforms into a surface integral at infinity via the divergence theorem, which strictly vanishes.

Thus, the variation of the interaction action reduces to:

$$\delta S_{\text{int}} = -2 \int \mathrm{d}^4x \, \xi_\nu (\partial_\mu T^{\mu\nu}) = 0$$

For the total action to remain gauge invariant, this variation must evaluate to zero for *any* arbitrary compactly-supported gauge parameter field $\xi_\nu(x)$. By the fundamental lemma of the calculus of variations, this integral vanishes identically if, and only if, the coefficient of ξ_ν is zero everywhere in $\mathcal{M}$:

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \tag{22}$$

Thus, the local gauge symmetry of the rank-2 tensor field strictly necessitates the conservation of the external energy-momentum tensor. Analogous to the real vector field case discussed in electrodynamics (cf. Exercise 1), identity (22) is an *off-shell* Noether identity representing a fundamental consistency condition of the theory.

2. To determine the classical equations of motion for the field $h_{\mu\nu}$ in the presence of the external source, the Principle of Least Action is employed. The dynamics are governed by the condition that the variation of the total action vanishes, $\delta S_{\text{total}} = \delta S + \delta S_{\text{int}} = 0$, for arbitrary symmetric variations $\delta h_{\mu\nu} \in C^2(\mathcal{M})$.

   The variation of the interaction action is straightforward:

$$\delta S_{\text{int}} = \int \mathrm{d}^4x \, (\delta h_{\mu\nu}) T^{\mu\nu} = \int \mathrm{d}^4x \, \delta h^{\mu\nu} T_{\mu\nu}$$

   For the free action S, each of the four terms in the Lagrangian density is varied individually. Application of integration by parts (moving the derivative from the variation δh to the field h), and discarding of the resulting total derivative boundary terms (by restricting $\delta h_{\mu\nu}$ to the subspace $C_c^2(\mathcal{M})$) yield the following:

   - **Term 1:**

$$\begin{aligned}\delta(\partial_\alpha h^{\kappa\lambda} \partial^\alpha h_{\kappa\lambda}) &= 2(\partial_\alpha \delta h^{\kappa\lambda}) \partial^\alpha h_{\kappa\lambda} \\ &\xrightarrow{\text{IBP}} -2\delta h^{\kappa\lambda} \Box h_{\kappa\lambda} = -2\delta h^{\mu\nu} \Box h_{\mu\nu}\end{aligned}$$

   - **Term 2:**

$$\begin{aligned}\delta(2\partial_\kappa h^{\kappa\lambda} \partial_\lambda h) &= 2(\partial_\kappa \delta h^{\kappa\lambda}) \partial_\lambda h + 2\partial_\kappa h^{\kappa\lambda} (\partial_\lambda \delta h) \\ &\xrightarrow{\text{IBP}} -2\delta h^{\kappa\lambda} \partial_\kappa \partial_\lambda h - 2\delta h \partial_\lambda \partial_\kappa h^{\kappa\lambda}\end{aligned}$$

     Using the trace variation $\delta h = \eta_{\mu\nu} \delta h^{\mu\nu}$ and relabeling indices, this becomes:

$$-2\delta h^{\mu\nu} \partial_\mu \partial_\nu h - 2\eta_{\mu\nu} \delta h^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta}$$

   - **Term 3:**

$$\begin{aligned}\delta(-2\partial_\alpha h^{\kappa\alpha} \partial^\lambda h_{\kappa\lambda}) &= -4(\partial_\alpha \delta h^{\kappa\alpha}) \partial^\lambda h_{\kappa\lambda} \\ &\xrightarrow{\text{IBP}} 4\delta h^{\kappa\alpha} \partial_\alpha \partial^\lambda h_{\kappa\lambda}\end{aligned}$$

Relabeling the contracted indices ($\kappa \to \mu$, $\alpha \to \nu$, $\lambda \to \alpha$) results in $4\delta h^{\mu\nu}\partial_\nu\partial^\alpha h_{\mu\alpha}$. Since the variation $\delta h^{\mu\nu}$ is symmetric, only the symmetric part of the contracted tensor ($\partial_\nu\partial^\alpha h_{\mu\alpha}$) is non-vanishing. Thus, explicit symmetrization of the operator over μ and ν results in:

$$\begin{aligned} 4\delta h^{\mu\nu}\partial_\nu\partial^\alpha h_{\mu\alpha} &= 4\delta h^{\mu\nu}\partial_{(\nu}\partial^\alpha h_{\mu)\alpha} \\ &= 2\delta h^{\mu\nu}(\partial_\nu\partial^\alpha h_{\mu\alpha} + \partial_\mu\partial^\alpha h_{\nu\alpha}) \end{aligned}$$

- **Term 4:**

$$\begin{aligned} \delta(-\partial_\alpha h\partial^\alpha h) &= -2(\partial_\alpha\delta h)\partial^\alpha h \\ &\xrightarrow{\text{IBP}} 2\delta h\Box h = 2\eta_{\mu\nu}\delta h^{\mu\nu}\Box h \end{aligned}$$

Summation of these contributions yields the total variation of the free action:

$$\begin{aligned} \delta S = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\,\delta h^{\mu\nu}\big[&-2\Box h_{\mu\nu} - 2\partial_\mu\partial_\nu h - 2\eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} \\ &+ 2(\partial_\mu\partial^\alpha h_{\nu\alpha} + \partial_\nu\partial^\alpha h_{\mu\alpha}) + 2\eta_{\mu\nu}\Box h\big] \end{aligned}$$

which is equivalent to:

$$\begin{aligned} \delta S = \frac{1}{8\pi G_N}\int \mathrm{d}^4x\,\delta h^{\mu\nu}\big[&\Box h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\nu\alpha} - \partial_\nu\partial^\alpha h_{\mu\alpha} \\ &+ \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\big] \end{aligned}$$

Imposing the condition $\delta S + \delta S_{\text{int}} = 0$ yields:

$$\begin{aligned} \int \mathrm{d}^4x\,\delta h^{\mu\nu}\Big\{\frac{1}{8\pi G_N}\big[&\Box h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\nu\alpha} - \partial_\nu\partial^\alpha h_{\mu\alpha} \\ &+ \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\big] + T_{\mu\nu}\Big\} = 0 \end{aligned}$$

For this integral to vanish for any arbitrary symmetric variation $\delta h^{\mu\nu}(x)$, the symmetric bracketed expression must vanish identically everywhere in $\mathcal{M}$ (by virtue of the restriction $\delta h_{\mu\nu} \in C_c^2(\mathcal{M})$, and the fundamental lemma of the calculus of variations), yielding the exact classical equations of motion:

$$\boxed{\Box h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\nu\alpha} - \partial_\nu\partial^\alpha h_{\mu\alpha} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} = -8\pi G_N T_{\mu\nu}} \tag{23}$$

The operator on the left-hand side equates to $-2G^{(1)}_{\mu\nu}$, where $G^{(1)}_{\mu\nu}$ is the *linearized* Einstein tensor, hence the given quadratic action reproduces the weak-field limit of General Relativity.

3. Simplification of the general equations of motion (23) is achieved by imposing the harmonic gauge condition, which is in fact a *class* of gauges, and is explicitly given by:

$$\partial^\alpha h_{\mu\alpha} - \frac{1}{2}\partial_\mu h = 0 \quad \Leftrightarrow \quad \partial^\alpha h_{\mu\alpha} = \frac{1}{2}\partial_\mu h$$

Substituting this constraint into the classical field equations (23), the mixed derivative terms are first evaluated:

- $-\partial_\mu(\partial^\alpha h_{\nu\alpha}) = -\partial_\mu\left(\frac{1}{2}\partial_\nu h\right) = -\frac{1}{2}\partial_\mu\partial_\nu h$
- $-\partial_\nu(\partial^\alpha h_{\mu\alpha}) = -\partial_\nu\left(\frac{1}{2}\partial_\mu h\right) = -\frac{1}{2}\partial_\nu\partial_\mu h = -\frac{1}{2}\partial_\mu\partial_\nu h$

Adding these to the existing $\partial_\mu\partial_\nu h$ term in the equation of motion gives:

$$-\frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\partial_\mu\partial_\nu h + \partial_\mu\partial_\nu h = 0$$

Thus, all *non-diagonal* second derivatives of the trace field are annihilated by the harmonic gauge. Evaluation of the double-divergence term $\eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}$ is in order. Application of the divergence operator ∂^μ to the harmonic gauge condition itself gives:

$$\begin{aligned}
\partial^\mu\left(\partial^\alpha h_{\mu\alpha} - \frac{1}{2}\partial_\mu h\right) &= 0 \\
\partial^\mu\partial^\alpha h_{\mu\alpha} - \frac{1}{2}\partial^\mu\partial_\mu h &= 0 \\
\partial_\alpha\partial_\beta h^{\alpha\beta} &= \frac{1}{2}\Box h
\end{aligned}$$

Substituting this relation into the remaining trace terms of the equation of motion gives:

$$-\eta_{\mu\nu}\Box h + \eta_{\mu\nu}\left(\frac{1}{2}\Box h\right) = -\frac{1}{2}\eta_{\mu\nu}\Box h$$

The complete equation of motion in the harmonic gauge reduces strictly to the trace-reversed wave equation:

$$\boxed{\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -8\pi G_N T_{\mu\nu}} \tag{24}$$

In this gauge, the differential operator becomes diagonal with respect to the derivatives (meaning there are no mixed partials coupling different components) analogous to the $\Box A_\mu = j_\mu$ result obtained using the Lorenz gauge in electrodynamics (cf. Exercise 1).

4. The trace-reversed equations of motion are transformed from position space to momentum space using the Fourier transform:

$$h_{\mu\nu}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x} h_{\mu\nu}(k), \qquad T_{\mu\nu}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x} T_{\mu\nu}(k)$$

Since $h_{\mu\nu}(k), T_{\mu\nu}(k)$ are functions of $k \in \mathcal{M}^*$ that belong in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ (thus, the interchange of the order of differentiation and integration, is allowed by the Lebesgue Dominated Convergence theorem), the differential equation (24) algebraically maps to:

$$-k^2 h_{\mu\nu}(k) + \frac{1}{2}\eta_{\mu\nu}k^2 h(k) = -8\pi G_N T_{\mu\nu}(k)$$

In order for the contravariant field tensor $h^{\alpha\beta}(k)$ to be factored out, the covariant components $h_{\mu\nu}(k)$ and the trace $h(k)$ must be written using the Minkowski metric. First, the trace is trivially written as the contraction:

$$h(k) = \eta_{\alpha\beta} h^{\alpha\beta}(k)$$

Second, in raising the indices of $h_{\mu\nu}(k)$, to ensure the resulting wave operator strictly inherits the intrinsic $\alpha \leftrightarrow \beta$ symmetry of the field $h^{\alpha\beta}$, an explicitly symmetrized identity for the Kronecker deltas is used:

$$\begin{aligned} h_{\mu\nu}(k) &= \frac{1}{2}\left(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} + \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\nu}\right) h_{\alpha\beta}(k) \\ &= \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right) h^{\alpha\beta}(k) \end{aligned}$$

Substituting these geometric identities back into the momentum-space equation yields:

$$-k^2\frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right) h^{\alpha\beta}(k) + \frac{1}{2}\eta_{\mu\nu}k^2\eta_{\alpha\beta}h^{\alpha\beta}(k) = -8\pi G_N T_{\mu\nu}(k)$$

Factoring out $h^{\alpha\beta}(k)$ and the common scalar $-\frac{1}{2}k^2$, it follows that:

$$-\frac{1}{2}k^2\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right) h^{\alpha\beta}(k) = -8\pi G_N T_{\mu\nu}(k)$$

By defining the rank-4 wave operator $D_{\mu\nu\alpha\beta}$ as the term in the brackets acting on the field, it follows that:

$$\boxed{D_{\mu\nu\alpha\beta} h^{\alpha\beta}(k) = -8\pi G_N T_{\mu\nu}(k)} \tag{25}$$

where the gauge-fixed wave operator is explicitly determined to be:

$$\boxed{D_{\mu\nu\alpha\beta} = -\frac{1}{2}k^2\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)} \tag{26}$$

The object in the parentheses is the so-called De Witt supermetric, which serves as the natural metric on the space of symmetric rank-2 tensors.

5. Since both $D_{\mu\nu\alpha\beta}$ and its inverse, $P^{\mu\nu\alpha\beta}$, belong to the space of endomorphisms on the space of symmetric rank-2 tensors, $\mathrm{End}(\mathrm{Sym}^2(\mathcal{M}))$, they must be linear combinations of the *invariant tensors*, which, by the First Fundamental theorem of Invariant Theory, form a basis of this linear space. By (26), *in the harmonic gauge*, the wave operator is a linear combination of tensor products of the Minkowski metric with itself (with the contribution of the momentum, k^μ, being restricted in a global scalar factor). Hence, the inverse wave operator, $P^{\mu\nu\alpha\beta}$, must belong in the span of the pure metric basis:

$$\{\eta^{\mu\nu}\eta^{\alpha\beta},\ \eta^{\mu\alpha}\eta^{\nu\beta},\ \eta^{\mu\beta}\eta^{\nu\alpha}\}$$

Since, by construction, $D_{\mu\nu\alpha\beta}$, is symmetric with respect to the first two ($\mu \leftrightarrow \nu$) and last two ($\alpha \leftrightarrow \beta$) indices, so must the inverse be. Imposing these two symmetry conditions on a general linear span of the above pure metric basis, it follows that the most general form for the inverse wave operator is:

$$P^{\mu\nu\alpha\beta} = c_1(k^2)\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) + c_2(k^2)\,\eta^{\mu\nu}\eta^{\alpha\beta} \tag{27}$$

for real coefficients $c_1(k^2), c_2(k^2)$ (that may depend on the Lorentz scalar $k^2 = k^\mu k_\mu$) that are to be determined. By definition, the contraction of the wave operator, $D_{\mu\nu\alpha\beta}$, with its inverse, $P^{\mu\nu\alpha\beta}$, must yield the projector to the subspace of *symmetric* rank-2 tensors, i.e.:

$$P^{\mu\nu\alpha\beta}D_{\alpha\beta\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho) \tag{28}$$

Substitution of (26) and (27) into (28) gives:

$$\begin{aligned}-c_1(k^2)\frac{k^2}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha})(\eta_{\alpha\rho}\eta_{\beta\sigma} + \eta_{\alpha\sigma}\eta_{\beta\rho} - \eta_{\alpha\beta}\eta_{\rho\sigma})\\ -c_2(k^2)\frac{k^2}{2}\eta^{\mu\nu}\eta^{\alpha\beta}(\eta_{\alpha\rho}\eta_{\beta\sigma} + \eta_{\alpha\sigma}\eta_{\beta\rho} - \eta_{\alpha\beta}\eta_{\rho\sigma}) = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho)\end{aligned}$$

from which, using the metric contraction $\eta^{\mu\alpha}\eta_{\alpha\rho} = \delta^\mu_\rho$, it follows that:

$$-k^2c_1(k^2)(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho) + k^2\left(c_1(k^2) + c_2(k^2)\right)\eta^{\mu\nu}\eta_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho)$$

Matching the coefficients in the last equality, which must hold for arbitrary $k^2 \in \mathbb{R}$, yields that:

$$c_1(k^2) = -c_2(k^2) = -\frac{1}{2k^2}\,, \qquad \forall\, k^2 \neq 0$$

Hence, the inverse of the wave operator, *in the harmonic gauge*, is given by:

$$\boxed{(D^{-1})^{\mu\nu\alpha\beta} \equiv P^{\mu\nu\alpha\beta} = -\frac{1}{2k^2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})\,, \quad \forall\, k^2 \neq 0} \tag{29}$$

which has the same De Witt supermetric tensor structure as the original operator in (26), as the De Witt supermetric happens to be an involution in a spacetime with dim $\mathcal{M} = 4$.

6. The solution for the field $h^{\mu\nu}(k)$ in momentum space is obtained straightforwardly by applying the inverse wave operator from the left on (25), and using (28):

$$h^{\mu\nu}(k) = -8\pi G_N P^{\mu\nu\alpha\beta} T_{\alpha\beta}(k)$$

Substituting (29), performing simple algebraic manipulation, exploiting the symmetry of the energy-momentum tensor, $T^{\mu\nu} = T^{\nu\mu}$ and lowering the indices, it follows that:

$$\boxed{h_{\mu\nu}(k) = \frac{8\pi G_N}{k^2}\left[T_{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu}T^{\rho}{}_{\rho}(k)\right]} \tag{30}$$

The solution for the field $h^{\mu\nu}(x)$ in spacetime is obtained by inverse Fourier transforming (30):

$$\boxed{h_{\mu\nu}(x) = 8\pi G_N \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2}\left[T_{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu}T^{\rho}{}_{\rho}(k)\right]} \tag{31}$$

which is the solution for the field $h^{\mu\nu}(x)$ produced by the (locally) conserved energy-momentum tensor $T^{\mu\nu}$ in the harmonic gauge.

It should be noted that the integral in (31) is singular at $k^2 = 0$ (on the *mass-shell*, i.e. the light cone), which is a manifestation of the fact that the harmonic gauge condition does *not* completely fix the gauge on the mass shell, since there is still a residual gauge freedom for gauge parameters $\xi_\mu(x)$ that satisfy the wave equation $\Box\,\xi_\mu(x) = 0$.

The momentum-space integral in (31) can be converted to a spacetime integral via the calculation of the Green function corresponding to the wave

operator (26). As in Exercise 1, the evaluation of the integral in (31) first involves the substitution of the Fourier transform of the source term, yielding:

$$h_{\mu\nu}(x) = -8\pi G_N \int \mathrm{d}^4 y \, G(x-y)\bar{T}_{\mu\nu}(y) \tag{32}$$

where:

$$G(x-y) = -\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2} \tag{33}$$

is the scalar Green's function for the d'Alembertian operator, and the trace-reversed source tensor:

$$\boxed{\bar{T}_{\mu\nu}(k) \coloneqq T_{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu}T^{\rho}{}_{\rho}(k)} \tag{34}$$

is defined for notational convenience. The evaluation of the Green's function (33) follows the same procedure that was outlined in Exercise 1, and, in fact, is exactly the same as the one for the d'Alembertian operator. The result is:

$$\boxed{G(x-y) = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|}\delta\left((x^0-y^0) - |\mathbf{x}-\mathbf{y}|\right)} \tag{35}$$

Substitution in (32), and integration of the δ-distribution, gives the final result:

$$\boxed{h_{\mu\nu}(t,\mathbf{x}) = -8\pi G_N \int \mathrm{d}^3\mathbf{y} \, \frac{\bar{T}_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}} \tag{36}$$

The result (36) is again the Retarded Potential formulae for the linearized gravitational field, explicitly stating that calculation of the gravitational field at position $\mathbf{x}$ and time t in a global Cartesian chart covering the entire flat Minkowskian manifold $\mathcal{M} = (\mathbb{R}^4, \eta)$, is the sum of the contributions of the "trace-reversed" source $\bar{T}_{\mu\nu}$ at all other spatial points $\mathbf{y}$, evaluated at a "retarded time" $t_{\mathrm{ret}} = t - r$ (where $r = |\mathbf{x}-\mathbf{y}|$ is the corresponding distance). It should be noted that the field $h_{\mu\nu}(x)$ is *not yet* associated with spacetime curvature (which is zero by assumption), but rather propagates *through* the flat Minkowski spacetime.

7. Given the momentum-energy tensor of a point particle of mass M static at the origin $\mathbf{x} = \mathbf{0}$:

$$T_{\mu\nu}(x) \equiv T_{\mu\nu}(t,\mathbf{x}) \equiv T_{\mu\nu}(\mathbf{x}) = M\,\delta^0_\mu\,\delta^0_\nu\,\delta^{(3)}(\mathbf{x}) \tag{37}$$

the momentum-space representation of the source is immediately obtained via a Fourier transform:

$$\begin{aligned} T_{\mu\nu}(k) &= \int \mathrm{d}^4k e^{-ik\cdot x} T_{\mu\nu}(x) = M\,\delta^0_\mu\,\delta^0_\nu \int \mathrm{d}^4k e^{-ik\cdot x}\delta^{(3)}(\mathbf{x}) \\ &= M\,\delta^0_\mu\,\delta^0_\nu \left(\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-ik_0 t}\right)\left(\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\delta^{(3)}(\mathbf{x})\right) \\ &= M\,\delta^0_\mu\,\delta^0_\nu \cdot (2\pi\delta(k_0)) \cdot (1) \end{aligned}$$

Hence, it follows that:

$$\boxed{T_{\mu\nu}(k) = 2\pi M\,\delta^0{}_\mu\,\delta^0{}_\nu\,\delta(k_0)} \tag{38}$$

Substitution of (38) into (31), after the trivial evaluation of the trace of the source $T_{\mu\nu}(k)$, yields that:

$$\begin{aligned} h_{\mu\nu}(x) &= 8\pi G_N \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2} 2\pi M\,\delta(k_0)\left[\delta^0{}_\mu\,\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right] \\ &= 8\pi G_N \int \frac{\mathrm{d}k_0\,\mathrm{d}^3\mathbf{k}}{(2\pi)^4}\frac{e^{ik_0 t}e^{-i\mathbf{k}\cdot\mathbf{x}}}{k_0^2 - |\mathbf{k}|^2}\, 2\pi M\delta(k_0)\left[\delta^0{}_\mu\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right] \\ &= -8\pi G_N M\left[\delta^0{}_\mu\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right]\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^2} \end{aligned}$$

where in the last line the sign of the exponent in the integrand is made positive, since the domain of integration is $\mathbb{R}^3$, and thus the sign of the exponent $\mathbf{k}\cdot\mathbf{x}$ is arbitrary under a parity transformation $\mathbf{k}\to-\mathbf{k}$. The last integral can be evaluated easily by exploiting spherical coordinates. The final result is:

$$h_{\mu\nu}(x) = -8\pi G_N M\left[\delta^0{}_\mu\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right]\left(\frac{1}{4\pi|\mathbf{x}|}\right) = -\frac{2G_N M}{|\mathbf{x}|}\left[\delta^0{}_\mu\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right]$$

The latter can be rewritten as:

$$\boxed{h_{\mu\nu}(x) \equiv h_{\mu\nu}(r) = 2\,\phi(r)\left[\delta^0{}_\mu\delta^0{}_\nu - \frac{1}{2}\eta_{\mu\nu}\right]} \tag{39}$$

where $r \equiv |\mathbf{x}|$ and the Newtonian gravitational potential:

$$\phi(r) = -\frac{G_N M}{r}\,, \qquad r \equiv |\mathbf{x}| \neq 0 \tag{40}$$

is defined. Writing the components of (39) explicitly, it follows that:

$$\boxed{h_{00}(x) = \phi(r), \quad h_{0i}(x) = 0, \quad h_{ij}(x) = \phi(r)\,\delta_{ij}, \quad i, j \in \{1, 2, 3\}} \tag{41}$$

which is the desired result. The static point mass generates a static, diagonal gravitational field which is governed by the Newtonian potential in both the temporal and spatial components.

It should be noted that the above analysis simply follows the statement of the exercise and to some extent is redundant, as, given the calculation of the Green's function (35), relation (36) can be employed immediately to give the result (39) without need for the intermediate calculation (38).

8. Thus far, the gravitational field $h_{\mu\nu}(x)$ is a dynamical field that propagates *through* the flat Minkowski spacetime $\mathcal{M} = (\mathbb{R}^4, \eta)$, and couples with energy-momentum external sources. The latter coupling, via the Weak Equivalence Principle, can be combined with the free kinetic term (dictated by $\eta_{\mu\nu}$) resulting in *free* motion in an effective geometry, with a metric tensor given by a perturbation of the Minkowski metric:

$$g_{\mu\nu}(r) \simeq \eta_{\mu\nu} + 2h_{\mu\nu}(r) \tag{42}$$

where the constant factor is inserted for normalization purposes. Equivalently, by (41), the perturbed metric has components:

$$g_{00}(r) \simeq 1 + 2h_{00}(r) = 1 + 2\phi(r), \quad g_{0i}(r) = g_{i0}(r) = 0 \tag{43}$$

$$g_{ij}(r) \simeq -\delta_{ij} + 2h_{ij}(r) = -\left[1 - 2\phi(r)\right]\delta_{ij} \tag{44}$$

By the general form for the infinitesimal line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, it is straightforward from the above that:

$$\mathrm{d}s^2 = g_{00}(r)\mathrm{d}t^2 + g_{ij}(r)\mathrm{d}x^i\mathrm{d}x^j \tag{45}$$

Thus, in Cartesian coordinates (t, x, y, z):

$$\boxed{\mathrm{d}s^2 \simeq \left[1 + 2\phi(r)\right]\mathrm{d}t^2 - \left[1 - 2\phi(r)\right](\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)} \tag{46}$$

where $r \equiv |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2} > 0$, and the Newtonian potential $\phi(r)$ is given by (40). In Spherical coordinates (t, r, θ, φ) the line element takes the following form:

$$\boxed{\mathrm{d}s^2 \simeq \left[1 + 2\phi(r)\right]\mathrm{d}t^2 - \left[1 - 2\phi(r)\right]\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right]} \tag{47}$$

as it can immediately be seen by transforming the line element (46) from Cartesian to spherical coordinates.

9. Substitution of the Newtonian potential (40) into (47) yields the following expression for the line element of the perturbed Minkowski metric:

$$\mathrm{d}s^2 \simeq \left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{2G_N M}{r}\right)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right] \tag{48}$$

The desired radial transformation, $R = R(r)$, should recover the weak-field Schwarzschild metric, i.e.:

$$\mathrm{d}s^2_{\mathrm{Sch}} \simeq \left(1 - \frac{2G_N M}{R}\right)\mathrm{d}t^2 - \left(1 + \frac{2G_N M}{R}\right)\mathrm{d}R^2 - R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{49}$$

in a first order approximation in $G_N M/r$. By inspection of (48) and (49), the coefficients of the angular parts of the line elements should be equal. Thus it follows that the desired transformation should be:

$$\boxed{R(r) \coloneqq r\left(1 + \frac{2G_N M}{r}\right)^{1/2}, \quad r > 0} \tag{50}$$

which, in first order in $G_N M/r$ (after Taylor expanding the radical), yields:

$$\boxed{R(r) \simeq r\left(1 + \frac{G_N M}{r}\right) = r + G_N M\,, \quad r \gg G_N M} \tag{51}$$

The transformation (50) inverted, $r = r(R)$, and inserted into (48), gives the line element in the new coordinates (t, R, θ, φ):

$$\mathrm{d}s^2 \simeq \left(1 - \frac{2G_N M}{r(R)}\right)\mathrm{d}t^2 - \left(1 + \frac{2G_N M}{r(R)}\right)\mathrm{d}r^2(R) + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{52}$$

The temporal and radial parts of (52) can be approximated to first order in $G_N M/r$ by inserting (51). The approximation scheme involves algebraic manipulation of the coefficients and Taylor expansions up to first order in $G_N M/r$, and is summarized in the following formulas:

$$\mathrm{d}r^2(R) \overset{(51)}{\simeq} \mathrm{d}R^2, \qquad r \gg G_N M \tag{53}$$

$$\frac{1}{r} \overset{(51)}{\simeq} \frac{1}{R - G_N M} = \frac{1}{R}\left(1 - \frac{G_N M}{R}\right)^{-1} \simeq \frac{1}{R}, \qquad r, R \gg G_N M \tag{54}$$

Substitution of the above pair of approximations into (52), yields:

$$\boxed{\mathrm{d}s^2 \simeq \mathrm{d}s^2_{\mathrm{Sch}}, \qquad r, R \gg G_N M} \tag{55}$$

as desired. It should be noted that, while the original isotropic radial coordinate r belongs to a *bookkeeping* chart with a locally conformally flat spatial geometry, the introduced areal radial coordinate R belongs to the *spherical shell* chart. By (51), it is apparent that $R > r$, hence the physical "areal" radius of a sphere drawn around a gravitating mass is larger than the coordinate distance measured from the center in flat space. Space itself is stretched inward toward the mass.

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